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FAN AND PUMP MODEL THAT HAS A UNIQUE SOLUTION FOR ANY PRESSURE BOUNDARY CONDITION AND CONTROL SIGNAL

Michael Wetter
Lawrence Berkeley National Laboratory
Environmental Energy Technologies Division
Building Technology and Urban Systems Department
Simulation Research Group, Berkeley, CA 94720, USA

ABSTRACT

Some building performance simulation programs compute the volume flow rate in ducts and pipe networks for a given fan speed, which may be computed by a feedback controller, based on the intersection of the flow resistance and fan curve. We show analytically and based on numerical simulations that fan models that use affinity laws and polynomials for the fan curve can lead to equations that become singular, have no, one, multiple or an infinite number of solutions. In experiments in which we used such a model, the simulations failed due to non-convergence or the time integration algorithm stalled due to oscillatory behavior. We therefore developed a new fan model, presented in this paper, that is a composition of a fan model that is used for low speeds and normal speeds. In each region, the model has a unique solution and eliminates the previously discussed numerical problems. The composite model is differentiable in all inputs. The approach for developing this model was to enforce constraints in each region that are mathematically sufficient to ensure the existence and uniqueness of a solution. The fan model is implemented in the free open-source Modelica Buildings library version 1.3.

INTRODUCTION

This paper describes a numerically robust implementation of a fan model that can be simulated for any speed and as a component of an arbitrary duct network. For brevity, we only write fan and ducts, but the discussions are also applicable for pumps and pipes. The motivation for this work is that affinity laws for fans are not applicable for system simulation with variable speed fans, as they yield singular equations as the speed goes to zero. Furthermore, polynomials that express the fan pressure raise as a function of the volume flow rate, as is customarily used in building performance simulation programs, can be shown to cause numerical problems: If such polynomials are combined with the pressure drop curve of the duct network, then the combined equation that is used to resolve the fan flow rate and associated pressure raise can have no, multiple or an infinite number of solutions. To complicate matters, existence of a solution cannot be checked statically for general polynomial fan curves. If a solution vanishes, a convergence error occurs during run-time.

It turns out that such polynomial equations and affinity laws are used in EnergyPlus 7.2 (EnergyPlus Engineering Reference, 2012) and in the Modelica Standard Library 3.2 (Modelica Association, 2010). CONTAM 2.4 (Walton and Dols, 2010) also uses polynomial equations but issues a warning if a user enters data that lead to an inflection point in the polynomial. Polynomial equations and affinity laws have also been used in the prerelease version 0.12 of the Modelica Buildings library (Wetter et al., 2013). In the meantime, they have been reimplemented in a robust way as described in this paper because experimental results revealed these convergence problems for which we present a mathematically sound reformulation in this paper.

Convergence problems with the implementation of fan curves based on polynomials and affinity laws have also been observed in other tools. For example, the EnergyPlus 7.2 Engineering Reference, p. 346, states that "the resolver [i.e., the solver that computes the flow and pressure drop distribution] finds the intersection of the two curves by successive substitution with 0.9 as a damping factor. If the flow rate is outside (or if in any iteration move out of) the range for which pump curve-fit is suggested, the resolver will bring the value within range..." This suggests that there are indeed convergence problems in EnergyPlus. On p. 343, the Engineering Reference states that "Pressure drop curves must not be placed on branches which only contain a pump." We show in this paper that for parallel pumps, or in our case parallel fans, such branches do require a pressure drop to avoid an infinite number of solutions.

CONVENTIONS AND NOMENCLATURE

1. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ the set of positive, non-zero real numbers, and by $\mathbb{N}$ the set of natural numbers $\mathbb{N} \triangleq \{1, 2, \ldots\}$.
2. We write $a \triangleq b$ to denote that a is equal to b by definition.
3. For $s, t \in \mathbb{R}$, we write $s \downarrow t$ to denote $s \to t$ with $s > t$. We write $s \approx t$ if $|s - t|$ is small. We write $s \propto t$ if s is proportional to t.
4. $f(\cdot)$ denotes a function where $(\cdot)$ stands for the undesignated variable. $f(x)$ denotes the value of $f(\cdot)$ for the argument x. $f\colon A \to B$ indicates that the domain of $f(\cdot)$ is in the space A, and that the image of $f(\cdot)$ is in the space B.

Throughout this paper, we will have to make the distinction between the solution to an equation and its nu-

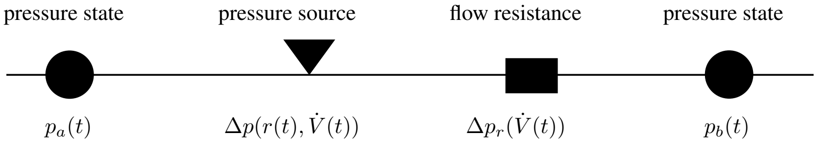

Figure 1: Schematic diagram for pressure boundary condition, pressure source of fan, and flow resistance.

merical approximation. Let $x \in \mathbb{R}^n$, for some $n \in \mathbb{N}$. We will write $x^*(\epsilon) \in \mathbb{R}^n$ to denote the numerical approximation to x obtained with solver precision control parameter $\epsilon \in \mathbb{R}_+$. For example, for $f(x) = 0$, $x \in \mathbb{R}^n$ is a solution to the equation. If obtaining this solution requires an iterative solver, then one computes an approximate solution $x^*(\epsilon) \in \mathbb{R}^n$ that satisfies, for some $K > 0$,

$$\|x - x^*(\epsilon)\| \leq K\,\epsilon. \tag{1}$$

PROBLEM DEFINITION

We consider the system shown in Figure 1 at some arbitrary time instant $t \in \mathbb{R}$. The boundary conditions of the system are an input signal for the normalized fan speed $r(t) \in [0,\, 1]$ and pressure states $p_a(t) \in \mathbb{R}_+$ and $p_b(t) \in \mathbb{R}_+$. The flow resistance can be described by a function $\Delta p_r \colon \mathbb{R} \times \mathbb{R}$ that maps volume flow rate to pressure drop. From fluid dynamics, we know that this function is continuously differentiable and strictly monotone increasing.[1] Without loss of generality, $\Delta p_r(\cdot)$ could also be a function of time, for example, in case of a controlled damper. The fan model implements an algebraic equation that computes the pressure raise $\Delta p\big(r(t), \dot V(t)\big)$. Fan curves that relate volume flow rate to pressure raise are not strictly decreasing if a fan has a stall region. However, below we will show that strict decrease is required to prove existence of a solution. A numerical solver then computes for some $r(t)$, $p_a(t)$ and $p_b(t)$ a numerical approximation to the flow rate $\dot V(t)$ that satisfies

$$0 = \Delta p(r(t), \dot V(t)) - \Delta p_r(\dot V(t)) + p_a(t) - p_b(t). \tag{2}$$

We are interested in how to formulate the fan curve $\Delta p(\cdot, \cdot)$ such that (2) has always a unique solution.

Now, continuous time system simulation programs can compute for any time instant $t \in \mathbb{R}$ and speed $r \in [0,\, 1]$, and for any solver tolerance $\epsilon \in \mathbb{R}_+$, an approximate solution $\dot V^*(\epsilon, t)$ to (2). In cases where no solution exists, the iterative solution will never converge, in case of multiple solutions, it may converge to an arbitrary solution and in cases of an infinite number of solutions, $\dot V^*(\epsilon, t)$ may show oscilattory behavior in time. Hence, a careful implementation that replaces the affinity laws and the polynomial equations for the fan curve with a more suited formulation is required for a robust simulation.

Note that even if the fan is controlled to be either on or off, the speed $r(t)$ may attain any value between 0 and 1. For example, a user may compute the speed as an output of a second order filter for two reasons: First, this provides a continuous change in flow rate, which can be faster and more robust to simulate compared to a step change. Second, the second order filter may be used to approximate the transient response of the fan. For this reason, the fan models in the Modelica Buildings library have a parameter that allows users to enable or disable such a second order filter.

FAN MODEL

This section explains how the volume flow rate versus pressure performance curve for a fan is implemented in the Modelica Standard Library 3.2, in the Buildings library 0.12, and in the Buildings library 1.3. The section also describes the numerical problems of the implementations in Modelica 3.2 and in Buildings 0.12. It then describes a new model, and proves that with the new model, a unique solution exists for the flow rate versus pressure relation if the model is part of a duct network.

Implementation in Modelica Standard Library 3.2

The Modelica Standard Library implements a model for a fan that computes the pressure rise as

$$\Delta p(r(t), \dot V(t)) = r(t)^2 \sum_{i=1}^{n} c_i \left(\frac{\dot V(t)}{r(t)} \right)^{i-1}, \tag{3}$$

where $r(t)$ is the normalized speed, defined as $r(t) = N(t)/N_0$, where $N(t)$ is the revolution, $\dot V(t)$ is the volume flow rate, $\{c_i\}_{i=1}^n$ are polynomial coefficients that are determined from user-provided operating points $\{\dot V_i,\, \Delta p_i\}_{i=1}^n$ at $r = 1$, and $\dot V(t)$ is the volume flow rate. This implementation is motivated by the fan similarity laws that state that $\Delta p \propto r^2$ and $\dot V \propto r$. However, this is problematic for three reasons:

1. Even if the term $N(t)/N_0$ in (3) were to be multiplied into the summation, the equation (3) is undefined for $r(t) = 0$ if $n \geq 4$.
2. Since the polynomial coefficients $\{c_i\}_{i=1}^n$ are determined by solving (3) for given flow rates and pressure rise $\{\dot V_i, \Delta p_i\}_{i=1}^n$ (for $r = 1$), there is no guarantee that (3) is monotone decreasing in the volume flow rate.

In the Modelica Standard Library, the first two problems are avoided by replacing $r(t)$ by

[1] Some building simulation programs, e.g., EnergyPlus, use $\Delta p_r(\dot V) = k\, \mathrm{sgn}(\dot V)\sqrt{|\dot V|}$ for some $K > 0$ even in a neighborhood around $\dot V = 0$. This should however be avoided as the first derivative is unbounded.

$\max(r(t), 0.001/N_0)$. However, this introduces a non-differentiability at $r(t) = 0.001/N_0$. Moreover, it causes $\Delta p(r(t), \dot{V}(t))$ to be non-zero even if $\dot{V}(t) = 0$ and $r(t) = 0$. Hence, the fan cannot be switched off completely, and volume flow occurs even if the HVAC system is off. This may introduce outside air when heaters are off, thereby causing subfreezing temperatures in heat exchangers which in turn can cause the simulation to stop. The third point has been shown to cause two solutions to exist for certain configurations of flow networks and fan curves. This caused non-physical results and divergence of the solver.

Implementation in Buildings Library 0.12

We will now explain how the first two problems were avoided in the Buildings library version 0.12. We will then explain why the new implementation was also not robust. Finally, after this section, we will explain how we reimplemented the fan model to circumvent all three problems.

In the Buildings library version 0.12, the first two problems are avoided by reformulating (3) as

$$\Delta p(r(t), \dot{V}(t)) = c_1\, r(t)^2 + c_2\, r(t)\, \dot{V}(t), \quad \text{(4a)}$$
$$\Delta p(r(t), \dot{V}(t)) = c_1\, r(t)^2 + c_2\, r(t)\, \dot{V}(t) + c_3\, \dot{V}^2(t), \quad \text{(4b)}$$
$$\Delta p(r(t), \dot{V}(t)) = \sum_{i=1}^{n} c_i\, r(t)^{n-i}\, \dot{V}^{i-1}(t), \quad \text{for } n \geq 4. \quad \text{(4c)}$$

This implementation has shown to be numerically problematic in a large system model in which the fan curve was modeled as a linear function.[2] The reason was that Dymola selected $\dot{V}(t)$ as an iteration variable, and computing $r(t)$ and $\Delta p(r(t), \dot{V}(t))$ required an iterative solution, i.e., they were approximated using some $r^*(\epsilon, t)$ and some $\Delta p^*(\epsilon, t)$. The governing equation was

$$\dot{V}^*(\epsilon, t) = \frac{\Delta p^*(\epsilon, r^*(\epsilon, t), \dot{V}^*(\epsilon, t)) - r^*(\epsilon, t)^2\, c_1}{r^*(\epsilon, t)\, c_2} \approx \frac{\Delta p^*(\epsilon, r^*(\epsilon, t), \dot{V}^*(\epsilon, t))}{r^*(\epsilon, t)\, c_2}. \quad (5)$$

Thus, computing $\dot{V}^*(\epsilon, t)$ required dividing numerical noise by numerical noise.
Moreover, since, in the system model, the fans were connected as shown in Figure 2, the fluid volumes of two fan models were coupled. Due to the model parameterization, multiple volumes were connected without a flow resistance in between, forming a sequence of connected volumes. Within this sequence, the mass flow rate became oscillatory and unstable, as shown in Figure 3, because at $r = 0$ (and $\Delta p = 0$), any value of $\dot{V}(t)$ satisfied the governing equation (4a) for the fan in the lower flow path. That is, there were an infinite number of solutions to (4a)! Not surprisingly, this eventually led, for the fan configuration shown in Figure 2, to the oscillatory behavior shown in Figure 3. Consequently, the solver stalled as it was required to make very small time steps to control the integration error of the conservation equations of the volumes that participated in the mass exchange shown in Figure 3.

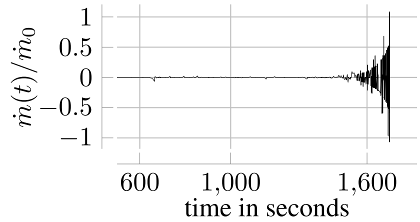

Figure 3: Instability of $\dot{m}(t)/\dot{m}_0$ for the fan inlet mass flow rate at location (a) in Figure 2.

We note that this oscillatory behavior is avoided in CONTAM 3.0 by replacing the fan model with an orifice model if the control signal satisfies $r(t) \leq \delta$ for some $0 < \delta < 1$. We did not use this approach as it would yield a hybrid model. Switching from one model to another leads to a state event that can increase the simulation time. Furthermore, if $r(t)$ is the output of a controller whose input depends on the fan volume flow rate, then the following problems can occur: For an algebraic hybrid model, there may not be a solution. For a dynamic hybrid model, the dynamics can introduce oscillatory behavior (chattering) which in turn can cause very slow progress of the time integration. As an example, consider the contaminant problem below that describes a volume with fresh air supply, constant contaminant source and feedback control on the fresh air supply flow rate. Let

$$V\, \frac{dC(t)}{dt} = \begin{cases} \dot{C}_s, & \text{if } u(t) < 0.2, \\ \dot{C}_s - \dot{V}\, C(t)\, u(t), & \text{otherwise,} \end{cases} \quad \text{(6a)}$$
$$C(0) = 0, \quad \text{(6b)}$$
$$u(t) = K_p\, C(t), \quad \text{(6c)}$$

where $V = 1\,\mathrm{m}^3$ is the control volume, $C(\cdot)$ is the contaminant concentration in $\mathrm{kg/m}^3$, $\dot{C}_s = 0.999\,\mathrm{kg/s}$ is a contaminant source, $K_p = 0.2\,\mathrm{m}^3/\mathrm{kg}$ is a control gain, $\dot{V} = 5\,\mathrm{m}^3/\mathrm{s}$ is the fan volume flow rate and $u(\cdot)$ is the fan control input. By (6a), the fan only operates if the control input signal satisfies $u(t) \geq 0.2$. The control law has no hysteresis.
When simulated in Dymola 2013 FD01, which has event detection, and the DASSL solver is used, which is an adaptive time step solver, $C(t)$ increases to $1\,\mathrm{kg/m}^3$ at $t = 1\,\mathrm{s}$, and then the control $u(t)$ chatters, causing the simulation to make very slow progress.

[2]This problem was observed in `https://corbu.lbl.gov/svn/bie/branches/mwetter/dev-zeroFlow/bie/modelica/Buildings`, revision 2721.

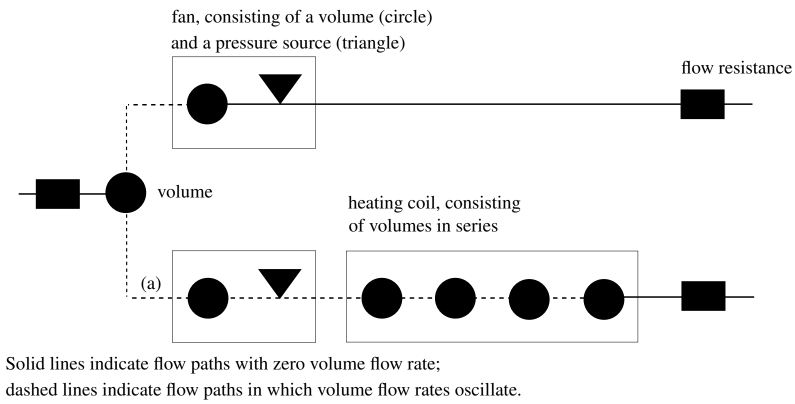

Figure 2: Schematic diagram of configuration of fans that caused unstable simulation.

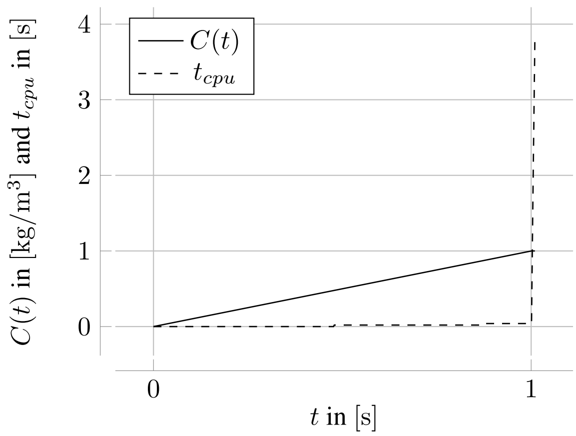

Figure 4: Contaminant concentration $C(t)$ and computing time t_{cpu} for equation (6).

This is illustrated in Figure 4. A test without hysteresis on the fan input signal that switches the fan on and off below a certain control signal, as in (6a), is also implemented in CONTAM 3.0.

Implementation in Buildings Library 1.3

To avoid the problems that we encountered in the Modelica Standard Library 3.2 and in the Buildings library 0.12, we reformulate the model as follows. To deduce a new fan model that always has a unique solution if it is part of a flow network, we first consider the thermodynamic system that contains the fan. Consider the system shown in Figure 1, where the pressures $p_a(t)$ and $p_b(t)$ are not a function of the volume flow rate.[3] For example, $p_a(t)$ may be the atmospheric pressure and $p_b(t)$ may be the room air pressure. The flow resistance $\Delta p_r(\dot{V}(t))$ may be composed of multiple branches. Without loss of generality, because the fan model $\Delta p(r(t),\, \dot{V}(t))$ has pressure only as an algebraic variable, any flow resistance between $p_a(t)$ and the pressure source of the fan can conceptually be lumped into the function $\Delta p_r(\cdot)$. For the configuration in Figure 1, we show in Figure 5 the volume flow rate

[3]However, their value can change as time progresses due to the flow exchanged with the control volume.

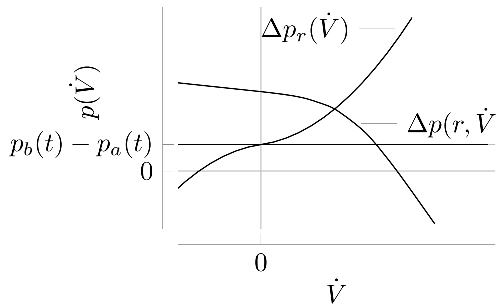

Figure 5: Pressure distribution in system shown in Figure 1.

versus the pressures.

This flow configuration has the following properties: Because the function for the flow resistance $\Delta p_r \colon \mathbb{R} \to \mathbb{R}$ is regularized near the origin, it follows that there exists a $\gamma > 0$ so that for all $\dot{V} \in \mathbb{R}$, either $\partial \Delta p_r(\dot{V})/\partial \dot{V} > \gamma$ or, for the special case of no flow resistance, $\partial \Delta p_r(\dot{V})/\partial \dot{V} = 0$. Furthermore, the difference of the state variables $p_b(t) - p_a(t)$ can have any sign.

To construct a fan model that has a unique solution for the volume flow rate, we use the following approach: First, we define a residual function that takes the volume flow rate $\dot{V}(t)$ as an independent variable. The speed $r(t)$, and the pressures $p_a(t)$ and $p_b(t)$ are assumed to be known when solving for $\dot{V}(t)$ as they are typically inputs or state variables of the system. We will construct this residual function so that it is continuously differentiable, and that the derivative is bounded away from zero. Continuity of the residual function, together with the bound on the derivative, ensures that the residual function eventually crosses zero for some $\dot{V}(t)$, which will be the solution to our volume flow rate versus pressure equation. Bounding the derivative away from zero also ensures that the residual function has no inflection points that may cause multiple solutions.

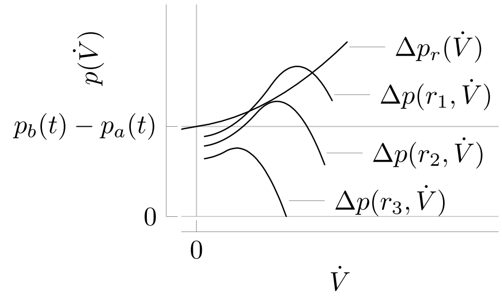

Figure 6: Flow resistance and fan curve that shows the situation where two solutions exist (with $\Delta p(r_1, \dot V)$) which collapse into one solution as $r_1 \to r_2$. This solution ceases to exist for $r_3 < r_2$.

Now, let

$$\Delta\widetilde{p}(r(t), \dot V(t)) \triangleq \Delta p(r(t), \dot V(t)) - \Delta p_r(\dot V(t)) + p_a(t) - p_b(t) \qquad (7)$$

be the residual function. Then, we seek a function $\Delta p\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is representative for the fan pressure rise, so that for any $r^* \in \mathbb{R}$, there exists a unique $\dot V^*$ that satisfies $\Delta\widetilde{p}(r^*, \dot V^*) = 0$. By continuity, a unique solution $\dot V^*$ exists if the derivative $\partial\Delta\widetilde{p}(r^*, \cdot)/\partial\dot V$ exists, is continuous and is bounded away from zero. Thus, a sufficient condition for a unique solution to exist is if, for any $r^* \in \mathbb{R}$, the function $\Delta\widetilde{p}(r^*, \cdot)$ has a continuous derivative and there exists a $\gamma > 0$ so that for all $\dot V \in \mathbb{R}$,

$$\frac{\partial\Delta\widetilde{p}(r^*, \dot V)}{\partial\dot V} < \gamma. \qquad (8)$$

Since for all $\dot V \in \mathbb{R}$,

$$\frac{\partial(-\Delta p_r(\dot V) + p_a - p_b)}{\partial\dot V} \leq 0, \qquad (9)$$

a sufficient condition is that for all $r \in \mathbb{R}$, the function $\Delta p(r, \cdot)$ has a continuous derivative and there exists a $\gamma > 0$ so that, for all $\dot V \in \mathbb{R}$,

$$\frac{\partial\Delta p(r, \dot V)}{\partial\dot V} < \gamma. \qquad (10)$$

The condition (10) will be used below to construct a model for a fan.

However, we first show that a solution may not exist if condition (10) is not satisfied. Consider Figure 6 which shows a fan with a stall region that works against a positive pressure. For $\Delta p(r_1, \dot V)$, there are two solutions. This situation can happen in a real system. If r_1 is reduced towards r_2, the locus of possible operating points move along the curve $\Delta p_r(\dot V)$, where they eventually collapse into one point. If r_2 is further reduced, this solution vanishes. In actual systems, the fan operating point will become unstable. However, the pressure dynamics needed to

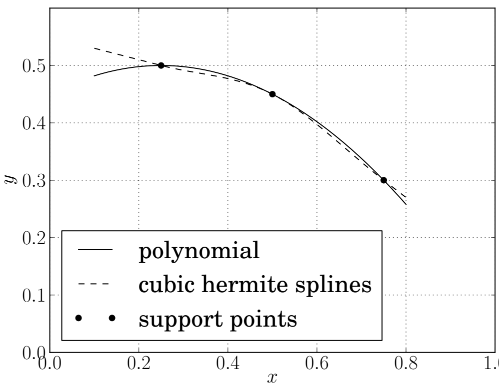

Figure 7: Fan curve obtained by using a polynomial and by using a cubic hermite spline.

represent this phenomenon is not modeled in a static fan model such as the one implemented here, and hence it cannot be expected that the solver will find the new solution.

Polynomials to represent fan pressure raise are commonly used in building simulation programs. Unfortunately, polynomials can have inflection points even if their coefficients are computed for support points that are strictly decreasing. For example, consider the 2nd order polynomial

$$y = 0.45 + 0.4\,x - 0.8x^2, \qquad (11)$$

obtained from the support points $\{(x_i,\, y_i)\}_{i=1}^3 = \{(0.25,\, 0.5),\, (0.5,\, 0.45),\, (0.75,\, 0.3)\}$. As shown in Figure 7, the support points are strictly decreasing, but the polynomial is increasing for small volume flow rates. Such an inflection point can be avoided by using cubic hermite splines that are computed using the algorithm of Fritsch and Carlson (1980). This algorithm guarantees strict descent of the cubic hermite spline if the data are strictly decreasing. Figure 7 shows such a cubic hermite spline that is computed for the above support points, and linearly extrapolated outside the range of the support points.

We will now create a fan model that satisfies condition (10) and the fan affinity laws in the typical region of operation. To simplify notation, let $\mathcal{S}_n = \{(\dot V_i, \Delta p_i)\}_{i=1}^n$, with $\dot V_i \geq 0$ and $\Delta p_i \geq 0$ for all $i \in \{1, \ldots, n\}$, denote the user-supplied performance data for the pressure rise between the fan inlet and outlet at full speed $r = 1$. Let $\delta = 0.05$, which we selected as a small number that is below the typical normalized fan speed.[4]
Then, we impose the following conditions:

1. For $r > \delta$, the affinity laws $\Delta p \propto r^2$ and $\dot V \propto r$ need to be satisfied.

[4]The value of $\delta = 0.05$ was chosen to be just below the typical normalized speed during the operation of the fan, which may be around 0.1 and 1.

2. For any $r \in \mathbb{R}$, there exists a $\gamma > 0$ so that
$$\frac{\partial \Delta p(r, \dot{V})}{\partial \dot{V}} < -\gamma, \tag{12a}$$
for all $\dot{V} \in \mathbb{R}$, if the points in $\mathcal{S}_n$ satisfy the descent condition
$$\frac{\Delta p_{i+1} - \Delta p_i}{\dot{V}_{i+1} - \dot{V}_i} < -\frac{\Delta p_{max}}{\dot{V}_{max}} \frac{\delta^2}{10}, \tag{12b}$$
for all $i \in \{1, \ldots, n-1\}$, where the maximum values are obtained by linear extrapolation as
$$\dot{V}_{max} = \dot{V}_n - \frac{\dot{V}_n - \dot{V}_{n-1}}{\Delta p_n - \Delta p_{n-1}} \Delta p_n, \tag{12c}$$
and
$$\Delta p_{max} = \Delta p_1 - \frac{\Delta p_2 - \Delta p_1}{\dot{V}_2 - \dot{V}_1} \dot{V}_1. \tag{12d}$$
3. For $r \leq \delta/2$, a different model is used that satisfies the following conditions:
   (a) If both, $r = 0$ and $\dot{V} = 0$, then $\Delta p(r, \dot{V}) = 0$.
   (b) $r = 0$ shall not imply that $\dot{V} = 0$.
   (c) If $r = 0$ and $\dot{V} > 0$, then $\Delta p(r, \dot{V}) < 0$ and conversely, if $r = 0$ and $\dot{V} < 0$, then $\Delta p(r, \dot{V}) > 0$.
4. For $r \in [\delta/2, \delta]$, the two models are combined so that the combined model is differentiable for any $r \in \mathbb{R}$.

Condition [2] ensures that there is only one intersection of the fan curve and the flow resistance curve. In equation (12b), the term on the right-hand side has been added to allow constructing a model that exactly reproduces the user-provided fan operating points, and that has an internal flow resistance that is needed to avoid an overspecified system of equations if $p_b(t) - p_a(t) \neq 0$ and $\Delta p_r(\dot{V}) = 0$ for all $\dot{V} \in \mathbb{R}$. Note that the correction of the descent condition (12b) due to the flow resistance is small, because for $\delta = 0.05$, we have $\delta^2/10 = 2.5 \cdot 10^{-4}$. Condition [3a] simply states that if the fan is off and has no flow, then it should have no pressure difference. Condition [3b] states that if the fan is off, there can be flow through the fan. Condition [3c] states that if the fan is off and there is flow, then the fan should have a flow resistance.

We implemented these conditions as follows: Let
$$\Delta\widehat{p}(\dot{V}(t)) = \dot{V}(t) \frac{\Delta p_{max}}{\dot{V}_{max}} \frac{\delta^2}{10}, \tag{13}$$
be a model of the flow resistance of the fan, approximated as a linear function of the volumetric flow rate. We define the fan model as the combination of a flow resistance and a pressure rise due to the fan revolution.

If the user provides two operating points ($n = 2$), then we set
$$\mathcal{S}'_n \triangleq \{(\dot{V}_i,\, \Delta p_i + \Delta\widehat{p}(\dot{V}_i)) \,|\, (\dot{V}_i,\, \Delta p_i) \in \mathcal{S}_n\}, \tag{14}$$
i.e., we correct the performance data for the fan-internal pressure drop $\Delta\widehat{p}(\cdot)$ in order for the fan to reproduce exactly the user-supplied performance data $\mathcal{S}_n$.
If $n > 2$, we add the points $(0, \Delta p_{max})$ and $(\dot{V}_{max}, 0)$ in order to ensure that the extrapolation of the fan performance curve is strictly decreasing. (If points for $\dot{V} = 0$ or $\Delta p = 0$ already exist in $\mathcal{S}_n$, then these points will not be added.) We set
$$\begin{aligned}\mathcal{S}'_n \;\triangleq\; & \{(0, \Delta p_{max})\} \\ & \cup\{(\dot{V}_i,\, \Delta p_i + \Delta\widehat{p}(\dot{V}_i)) \,|\, (\dot{V}_i,\, \Delta p_i) \in \mathcal{S}_n\} \\ & \cup\{(\dot{V}_{max}, 0)\}.\end{aligned} \tag{15}$$
With this construction, if the points in $\mathcal{S}_n$ satisfy the descent condition (12b), then the points in $\mathcal{S}'_n$ are strictly decreasing. It should now be clear that because of the addition of the term $\Delta\widehat{p}(\dot{V}_i)$ in (14) and (15), we added the term on the right-hand side in (12b).

For $r(t) > \delta$, we define this combined equation as follows: First, let $h(\cdot, \mathcal{S}'_n)$ be a cubic hermite spline that maps volume flow rate to pressure rise. Let $h(\cdot, \mathcal{S}'_n)$ be strictly decreasing if $\mathcal{S}'_n$ defines a strictly decreasing sequence. This is achieved by using the algorithm of Fritsch and Carlson (1980), which guarantees strict descent of the cubic hermite spline if the data are strictly decreasing. For $\dot{V}(t) \notin [0, \dot{V}_{max}]$, we linearly extrapolate the cubic hermite spline. For $r(t) > \delta$, we define the performance curve as
$$\begin{aligned}\Delta p^+(r(t), \dot{V}(t)) \;=\; & -\Delta\widehat{p}(\dot{V}(t)) \\ & + r(t)^2\, h\big(\dot{V}(t)/r(t), \mathcal{S}'_n\big).\end{aligned} \tag{16}$$
To see that for all $r > \delta$, there exists a $\gamma > 0$ so that for all $\dot{V} \in \mathbb{R}$, the expression $\Delta p^+(r(t), \dot{V}(t))$ satisfies
$$\frac{\partial \Delta p^+(r, \dot{V})}{\partial \dot{V}} < -\gamma, \tag{17}$$
we note that without loss of generality, the term $-\Delta\widehat{p}(\dot{V}(t))$ can be absorbed in $\Delta p_r(\dot{V}(t))$ since it is also a resistance. Hence, we require that
$$\frac{\partial (r(t)^2\, h\big(\dot{V}(t)/r(t), \mathcal{S}'_n\big)}{\partial \dot{V}} < -\gamma. \tag{18}$$
From condition (12b) follows that the points $\mathcal{S}'_n$ define a strictly decreasing sequence and hence $h\big(\dot{V}(t)/r(t),\, \mathcal{S}'_n\big)$ is also strictly decreasing due the algorithm of Fritsch and Carlson. Furthermore, since $h(\cdot, \mathcal{S}'_n)$ is linearly extrapolated using the derivatives at $\dot{V} = 0$ or $\dot{V} = \dot{V}_{max}$, which both are nonzero due to $h(\cdot, \mathcal{S}'_n)$ being strictly decreasing on the compact

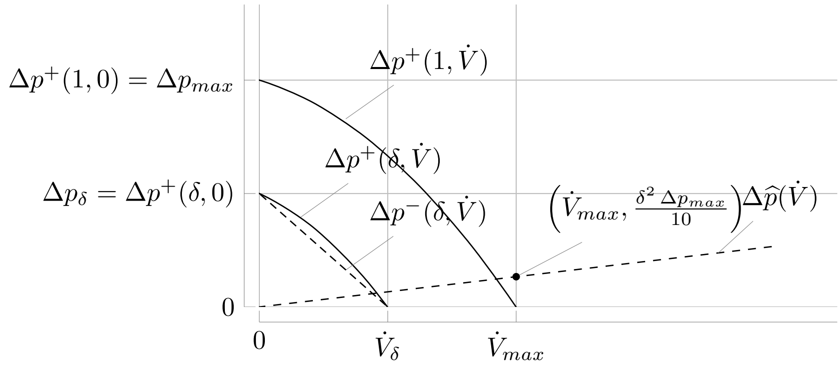

Figure 8: Construction of fan model near origin. (For better visibility, δ is not to scale.)

interval $\mathcal{I} = [0,\, \dot V_{max}]$, there exists a $\dot V^* \in \mathcal{I}$ so that $\partial h(\dot V^*, \mathcal{S}'_n)/\partial \dot V \geq \partial h(\dot V, \mathcal{S}'_n)/\partial \dot V$ for all $\dot V \in \mathcal{I}$. Hence, we can pick $\gamma = 1/2\, \partial h(\dot V^*, \mathcal{S}'_n)/\partial \dot V$. This proves that (18) is satisfied.

For $r(t) \leq \delta$, we proceed as follows: Define $\Delta p_\delta = \Delta p^+(\delta, 0)$ and $\dot V_\delta$ as the solution to $0 = \Delta p^+(\delta, \dot V_\delta)$, which is $\dot V_\delta = \delta\, \dot V_{max}$, because

$$\begin{aligned} 0 &= \Delta p^+(\delta, \dot V_\delta) = \delta^2\, h(\dot V_\delta/\delta,\, \mathcal{S}'_n) \\ &= h(\dot V_\delta/\delta,\, \mathcal{S}'_n) = h(\dot V_{max},\, \mathcal{S}'_n). \end{aligned} \tag{19}$$

Next, for $r < \delta/2$, we define the fan curve

$$\begin{aligned} \Delta p^-(r(t), \dot V(t)) &= r(t)\, \Delta p_\delta + r(t)^2\, (\bar c_1 + \bar c_2\, \dot V(t)) \\ &\quad - \Delta\widehat{p}(\dot V(t)), \end{aligned} \tag{20}$$

where $\bar c_1$ and $\bar c_2$ are obtained from the linear system of equations

$$0 = \Delta p^-(\delta, \dot V_\delta), \tag{21a}$$

$$\Delta p_\delta = \Delta p^-(\delta, 0), \tag{21b}$$

from which follows that $\bar c_1 > 0$ and $\bar c_2 < 0$.
With this construction, we obtain

$$\frac{\partial \Delta p^-(r(t), \dot V(t))}{\partial \dot V} = r(t)^2\, \bar c_2 - \frac{\Delta p_{max}}{\dot V_{max}} \frac{\delta^2}{10} < 0, \tag{22}$$

for all $r(t) \in \mathbb{R}$ and $\dot V(t) \in \mathbb{R}$, because $\bar c_2 < 0$ by construction. Note that the derivative in (22) does not depend on $\dot V(t)$ and hence for all $r(t) \in \mathbb{R}$, there exists a $\gamma > 0$ so that $\partial \Delta p^-(r(t), \dot V(t))/\partial \dot V < -\gamma$ for all $\dot V(t) \in \mathbb{R}$.

Remark 1.1 In (20), the term $r(t)\, \Delta p_\delta$ has been added to ensure that

$$\frac{\partial \Delta p^-(r(t), 0)}{\partial r} = \Delta p_\delta + 2\, r(t)\, \bar c_1 > 0, \tag{23}$$

for all $r(t) > -\Delta p_\delta/(2\, \bar c_1)$. In our experience, adding this term yields faster convergence when the fan is off. □

Finally, for the region $r \in [\delta/2, \delta]$, we combined the two functions using

$$\begin{aligned} \Delta p(r(t), \dot V(t)) &= \mathcal{R}\big(r(t) - \frac{3}{4}\,\delta, \\ &\qquad \Delta p^+\big(r(t), \dot V(t)\big), \\ &\qquad \Delta p^-\big(r(t), \dot V(t)\big),\ \frac{\delta}{4}\big), \end{aligned} \tag{24}$$

where $\mathcal{R}\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is a once continuously differentiable function, defined as

$$\mathcal{R}(x, y_1, y_2, \delta) = \begin{cases} y_1, & \text{if } x > \delta, \\ y_2, & \text{if } x < -\delta, \\ \frac{x}{\delta}\left(\left(\frac{x}{\delta}\right)^2 - 3\right) \frac{y_2 - y_1}{4} + \frac{y_1 + y_2}{2}, & \text{otherwise.} \end{cases} \tag{25}$$

In Modelica, this function is called `regStep` and provided by the `Modelica.Fluid` library. This construction yields a once continuously differentiable function for the fan pressure raise for the whole domain of operation.

We can now make the following important conclusion:

Remark 1.2 Let the fan pressure rise be defined by (24) and let the fan be a component of the system that is shown in Figure 1, with volume flow rate versus pressure rise performance data $\mathcal{S}_n$ that satisfy the decrease condition (12b). Then, we proved that for any given $r(t) \in \mathbb{R}$ and any given pressure difference $p_b(t) - p_a(t)$, there exists a *unique* solution to the flow versus pressure relation. Furthermore, the residual function (7) is once continuously differentiable and its derivative is bounded away from zero. Therefore, it is possible to reliably solve for the volume flow rate or the pressure drop.
If the points in $\mathcal{S}_n$ do not satisfy the decrease condition (12b), then multiple solutions can exist, and solutions can vanish as $r(t) \downarrow 0$. This can, of course, cause a numerical solver to fail. In our implementation of the fan model, we allow the points in $\mathcal{S}_n$ to violate the decrease condition (12b) as users may want to simulate

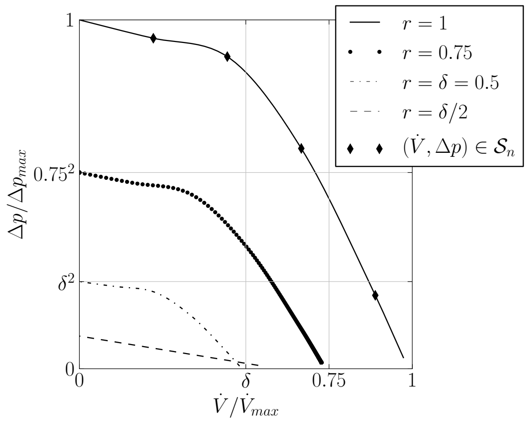

Figure 9: Fan volume flow rate versus pressure rise for different values of r*. For better illustration, we set* δ *to a large value of* 0.5 *instead of the more typical value of* 0.05.

the stall region, but for these parameters, the model will write a warning prior to the simulation. □

Figure 9 shows the volume flow rate versus pressure rise for different values of r according to (24). Note that for $r = \delta/2$, the performance curve is linear in the volume flow rate. The diamonds mark the four user-provided support points. The curves for $r \in \{1, 0.75, \delta\}$ do not exactly intersect at $(\dot{V}/\dot{V}_{max}, \Delta p/\Delta p_{max}) \in \{(1,0), (0.75,0), (\delta,0)\}$ because of the contribution of $\Delta\widehat{p}(\dot{V})$ that emulates the internal flow resistance of the fan. This is of no concern because the difference is small and the affinity laws are an idealization that do not take into account the flow resistance of the fan casing.

SUMMARY

We showed in Figure 6 and in Figure 7 that the widely used representation of a fan curve as a polynomial is an unfortunate choice as even for descending support points, the resulting fan curve can have an inflection point that causes numerical problems. To remedy this problem, we present a mathematically better suited representation of the fan curve. We also showed analytically by (5) and experimentally by Figure 3 that fan models that lack a flow resistance can cause the simulation to produce unrealistic flow rates and to fail. Such a flow resistance is therefore built into our fan model. As affinity laws lead to a singularity near zero speed, we present another formulation for low speeds that does not suffer from this problem. This led to two models, one for low speeds and one for normal speeds. In (12b), we present a sufficiency condition that can be tested prior to simulation to see whether a unique point of intersection between the fan curve and an arbitrary flow resistance curve, possibly with an added static pressure, exists for both of these models. In (6), we showed that a hybrid fan model can lead solvers to stall when used in conjunction with a feedback control loop. Therefore, we presented a formula that combines these two models in such a way that the combined fan model is differentiable, and hence avoids the need for a hybrid model.

CONCLUSIONS

Textbook equations for thermofluid systems are, in general, not applicable for system simulations. Nevertheless, they are used in many building simulation programs. However, it is well known that solvers in building simulation programs often fail to converge if models are large. In view of the singularities that can arise if textbook equations are used in a system simulation program, and due to the lack of rigor in implementing numerical solvers, this should not be a surprise. Mathematics cannot be fooled. To implement model equations such that the resulting coupled system of equations can be solved robustly, it is important that equations are implemented with care to avoid (i) singularities, (ii) solutions that vanish or (iii) multiple solutions for certain boundary conditions. The current practice in many building simulation programs is that values that do not satisfy the model equations are accepted if convergence is not achieved, and the simulation proceeds. Wetter and Wright (2004) showed that this leads to undesirable effects.

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